Mathematical Fragments for Quantum Optics

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1 Introduction

Quantum Optics builds on most of the machinery developed in basics undergraduate *Quantum Mechanics* courses. While out of the scope of this manuscript, we will start by stating (and demonstrating) the results that are most relevant for Quantum Optics.

1.1 Useful Formulas and Identities

In this section, we will state the most useful identities and formulas that ofter appear in Quantum Optics problems

1.1.1 Quantum States

Formula 1.1 (Operator Uncertainty)**.** *Given an operator $\hat{x}$, its variance is given by*

$$\Delta x^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2 . \tag{1.1}$$

Formula 1.2 (Uncertainty Principe)**.** *Given two operators $\hat{x}$ and $\hat{y}$, their uncertainties (or the square root of their variances) is given by*

$$\Delta x \Delta y \geq \frac{1}{2} \left| \langle [\hat{x}, \hat{y}] \rangle \right| \tag{1.2}$$

Formula 1.3 (Displacement Operator)**.** *Given a quantum state, expressed in the Fock basis $\{|n\rangle\}$, the displacement operator is written as*

$$\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^\star \hat{a}}, \tag{1.3}$$

where $\alpha \in \mathbb{C}$ defines the displacement both in momentum and position.

Formula 1.4 (Squeezing Operator)**.** *Given a quantum state, expressed in the Fock basis $\{|n\rangle\}$, the squeezing operator is written as*

$$\hat{S}(\xi) = e^{\left[\frac{1}{2}\xi^\star (\hat{a})^2 - \xi (\hat{a}^\dagger)^2\right]} \tag{1.4}$$

where $\xi r e^{i\theta}$ defines the squeezing direction θ and squeezing strength r.

1.1.2 Operator Manipulations

Formula 1.5 (Pauli Matrices)**.** *The Pauli matrices, particularly useful to study two-level systems, are given by*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.5}$$

Formula 1.6 (Matrix Exponential)**.** *The matrix exponential of an Operator A follows direclty from the Taylor expansion of the exponential function*

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \mathbb{I} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots \tag{1.6}$$

Formula 1.7 (Baker-Campbell-Hausdorff)**.** *The Baker-Campbell-Hausdorff formula expresses the exponentiation of two non-commuting operators:*

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\cdots}. \tag{1.7}$$

Formula 1.8 (Special Baker-Campbell-Hausdorff)**.** *It is often useful, particularly in the Schrieffer-Wolff transformation, to consider the following identity:*

$$e^S A e^{-S} = A + [S,A] + \frac{1}{2!}[S,[S,A]] + \frac{1}{3!}[S,[S,[S,A]]] + \dots \tag{1.8}$$

1.1.3 Dissipative Quantum Systems

Formula 1.9 (Master Equation in Lindblad Form)**.** *An open quantum systems described by the density operator ρ evolves according to the master equation, namely*

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[\hat{H},\rho] + \sum_i \frac{\gamma_i}{2}\left[2\hat{L}_i\rho\hat{L}_i^\dagger - \{\rho,\hat{L}_i^\dagger\hat{L}_i\}\right] \tag{1.9}$$

Formula 1.10 (Optical Bloch Equations)**.** *Consider a two level system driven off resonantly with $\delta = \omega_L - \omega_{eg}$ with Rabi frequency Ω, while undergoing spontaneous population decay Γ and total coherence decay $\gamma = \gamma_\mathrm{d} + \frac{\Gamma}{2}$, where γ_d is the pure dephasing rate. Its time evolution is give by:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\langle\sigma_{eg}(t)\rangle\\ -\langle\sigma_z(t)\rangle\\ \langle\sigma_{ge}(t)\rangle\end{pmatrix} = \begin{pmatrix}-(\gamma-i\delta) & \frac{i\Omega}{2} & 0\\ i\Omega & -\Gamma & -i\Omega\\ 0 & -\frac{i\Omega}{2} & -(\gamma+i\delta)\end{pmatrix}\begin{pmatrix}\langle\sigma_{eg}(t)\rangle\\ -\langle\sigma_z(t)\rangle\\ \langle\sigma_{ge}(t)\rangle\end{pmatrix} + \begin{pmatrix}0\\ -\Gamma\\ 0\end{pmatrix}. \tag{1.10}$$

Formula 1.11 (Quantum Regression Theorem)**.** *Let us consider a set of time-varying operators $\{\hat{x}_i\}$. If, the expectation value of the time varying operators satisfy the relationship*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}_i(t)\rangle = M_{ij}\langle\hat{x}_j(t)\rangle + \lambda_i, \tag{1.11}$$

then the following relationship is also valid

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}_l(t)\hat{x}_i(t+\tau)\hat{x}_k(t)\rangle = M_{ij}\langle\hat{x}_l(t)\hat{x}_j(t+\tau)\hat{x}_k(t)\rangle + \lambda_i\langle\hat{x}_l(t)\hat{x}_k(t)\rangle. \tag{1.12}$$

The initial conditions for the differential equations above are naturally given by the term

$$\langle\hat{x}_l(t)\hat{x}_i(t+\tau)\hat{x}_k(t)\rangle, \tag{1.13}$$

which together with the $\langle\hat{x}_l(t)\hat{x}_k(t)\rangle$ can be found from the steady-state solutions of the optical bloch equations, assuming that the fields are non-dynamical.

1.1.4 Field Statistics

Formula 1.12 (Electric Field Operator)**.** *The electric field operator is given by*

$$E^{(+)}(\vec{r},t)=\sum_k\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{\frac{1}{2}}\xi_k\,\hat{a}_k\,e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}. \tag{1.14}$$

Formula 1.13 (Electric Field Correlation Function)**.**

$$g^{(1)}(\tau)=\frac{G^{(1)}(\tau)}{G^{(1)}(0)}=\frac{\left\langle E^{(-)}(t)E^{(+)}(t+\tau)\right\rangle}{\left\langle E^{(-)}(t)E^{(+)}(t)\right\rangle} \tag{1.15}$$

Formula 1.14 (Wiener–Khinchin theorem)**.** *The emission spectrum $\mathcal{S}(\omega)$ is given by the Fourier transform of the field correlation function, namely*

$$\mathcal{S}(\omega)=\mathcal{F}\left\{G^{(1)}(\tau)\right\}. \tag{1.16}$$

Formula 1.15 (Intensity Correlation Function)**.** *Assuming the field is stationary*

$$g^{(2)}(\tau)=\frac{G^{(2)}(\tau)}{G^{(1)}(t;0)G^{(1)}(t+\tau;0)}=\frac{\left\langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\right\rangle}{\left\langle E^{(-)}(t)E^{(+)}(t)\right\rangle^2} \tag{1.17}$$

1.2 Time Evolution of Pure States

We know that a quantum system is fully described by its state vector $|\psi\rangle\in\mathcal{H}$. The time evolution of the state vector is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle=\hat{H}(t)\left|\psi(t)\right\rangle, \tag{1.18}$$

where $\hat{H}(t)\in\mathcal{H}^2$ is the Hamiltonian of the system.

Lemma 1.16. *The time evolution according to the Schrödinger equation preserves the norm of the state vector $\|\psi(t)\|$.*

Proof. We begin by considering the time derivative of the norm squared

$$\begin{aligned}\frac{\partial}{\partial t}\|\psi(t)\|^2&=\frac{\partial}{\partial t}\left\langle\psi(t)|\psi(t)|\psi(t)|\psi(t)\right\rangle\\&=\frac{\partial}{\partial t}\left\{\langle\psi(t)|\right\}|\psi(t)\rangle+\langle\psi|\frac{\partial}{\partial t}\left\{|\psi(t)\rangle\right\}.\end{aligned}$$

From 1.18 and its Hermitian conjugate we get that

$$\begin{aligned}\frac{\partial}{\partial t}\|\psi(t)\|^2&=-\frac{1}{i\hbar}\left\langle\psi(t)\right|\hat{H}^\dagger\left|\psi(t)\right\rangle+\frac{1}{i\hbar}\left\langle\psi(t)\right|\hat{H}\left|\psi(t)\right\rangle\\&=-\frac{1}{i\hbar}\left\langle\psi(t)\right|\hat{H}\left|\psi(t)\right\rangle+\frac{1}{i\hbar}\left\langle\psi(t)\right|\hat{H}\left|\psi(t)\right\rangle=0,\end{aligned}$$

where we have used the fact that the Hamiltonian, being an observable, is hermitian (i.e. $\hat{H}^\dagger = \hat{H}$). Thus we concluded that the norm does not vary with time. □

From Lemma 1.16, we conclude that the state vector at time t, is given by a unitary transformation $\hat{U}(t) \in \mathcal{H}^2$ acting on the state vector at time $t = 0$, namely

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle\,. \tag{1.19}$$

In the context of time evolution, the unitary transformation $\hat{U}(t)$ is called the ***time propagator***.

1.2.1 Time Propagator

Let us now find the for of the time propagator for a generic system having the Hamiltonian $\hat{H}(t)$. To this end we consider equation 1.18, by rearranging its term and integrating both sides we get

$$\begin{aligned}
\frac{\mathrm{d}\,|\psi(t)\rangle}{|\psi(t)\rangle} &= -\frac{i}{\hbar}\hat{H}(t)\,\mathrm{d}t \\
\int_{|\psi(0)\rangle}^{|\psi(\tau)\rangle} \frac{\mathrm{d}\,|\psi(t)\rangle}{|\psi(t)\rangle} &= -\frac{i}{\hbar}\int_0^\tau \hat{H}(t)\,\mathrm{d}t \\
|\psi(\tau)\rangle &= e^{-\frac{i}{\hbar}\int_0^\tau \hat{H}(t)\,\mathrm{d}t}\,|\psi(0)\rangle\,.
\end{aligned}$$

By substituting $\tau \to t$ and $t \to t'$ in the expression above we get

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\int_0^t \hat{H}(t')\,\mathrm{d}t'\right)|\psi(0)\rangle\,. \tag{1.20}$$

Comparing equations 1.19 and 1.20 yield to the general form of the time propagator for a time dependent Hamiltonian

$$\hat{U}(t) = \exp\left(-\frac{i}{\hbar}\int_0^t \hat{H}(t')\,\mathrm{d}t'\right)|\psi(0)\rangle\,. \tag{1.21}$$

We note that if $\hat{H}$ is not explicitly dependent on time, its associated time propagator reduces to $\hat{U}(t) = \exp\left(-i\hat{H}t/\hbar\right)$, and therefore the state vector evolves as

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}\,|\psi(0)\rangle\,. \tag{1.22}$$

1.2.2 Time Evolution of Stationary States

Let us now consider the eigenstates of the Hamiltonian $\{|E_i\rangle : i = 0, 1, 2, \ldots, N\}$. Such states are called *stationary states* (we will briefly see why). By definition we have

$$\hat{H}\,|E_i\rangle = E_i\,|E_i\rangle\,, \tag{1.23}$$

where E_i is the energy associated to the state $|E_i\rangle$.

We now want to study how stationary states evolve under the Schrödinger equation. To this end, we assume that the system is initially in state $|E_n\rangle$ and we write

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}} |E_i\rangle . \tag{1.24}$$

Note that here we have considered a time-independent Hamiltonian. While simplifying things, a similar argument can be made for the time-dependent case.

We continue by considering the definition of the *matrix exponential* for a generic operator $\hat{A} \in \mathcal{H}^2$ given by

$$e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{\hat{A}^n}{n!} = \mathbb{I} + \hat{A} + \frac{\hat{A}^2}{2!} + \frac{\hat{A}^3}{3!} + \dots . \tag{1.25}$$

We now apply 1.25 to equation 1.24. We therefore get

$$\begin{aligned} |\psi(t)\rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{it}{\hbar}\right)^n \hat{H}^n |E_i\rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{it}{\hbar}\right)^n E_i^n |E_i\rangle \\ &= \sum_{n=0}^{\infty} \frac{\left(-\frac{iE_i t}{\hbar}\right)^n}{n!} |E_i\rangle = e^{-\frac{iE_i t}{\hbar}} |E_i\rangle . \end{aligned}$$

We conclude that the eigenstates of the Hamiltonian $|E_i\rangle$ evolve by acquiring a phase dictated by their energy E_i. For this reason, they are called *stationary states*.

1.2.3 Time Evolution of Generic States

The results outlined in Section 1.2.2 are particularly useful when trying to describe the evolution of a generic state.

To this end, we assume a system having a time-independent Hamiltonian $\hat{H}$ with eigenstates $\{|E_i\rangle : i = 0, 1, 2, \dots, N\}$ and its associated eigenvaules such that equation 1.23 is valid. In addition we assume that the system at time $t = 0$ is in the state $|\psi(0)\rangle$. From the *spectral theorem*, we know we can write

$$|\psi(0)\rangle = \sum_{i=0}^{N} C_i |E_i\rangle , \tag{1.26}$$

where $C_i \in \mathbb{C}$ are coefficients that can be determined by multiplying both sides

of the equation above by $\langle E_j|$ as in

$$\begin{aligned}\langle E_j|\psi(0)|E_j|\psi(0)\rangle &= \langle E_j| \sum_{i=0}^{N} C_i \, |E_i\rangle \\ &= \sum_{i=0}^{N} C_i \, \langle E_j|E_i|E_j|E_i\rangle \\ &= \sum_{i=0}^{N} C_i \delta_{ij} = C_j.\end{aligned}$$

Therefore, combining the result above with equation 1.26, we get that

$$|\psi(0)\rangle = \sum_{i=0}^{N} \langle E_i|\psi(0)|E_i|\psi(0)\rangle \, |E_i\rangle \, . \tag{1.27}$$

To find the expression for $|\psi(t)\rangle$ we can apply the time propagator for time independent Hamiltonians, as in

$$\begin{aligned}|\psi(t)\rangle &= e^{-\frac{i\hat{H}t}{\hbar}} \sum_{i=0}^{N} \langle E_i|\psi(0)|E_i|\psi(0)\rangle \, |E_i\rangle \\ &= \sum_{i=0}^{N} \langle E_i|\psi(0)|E_i|\psi(0)\rangle \, e^{-\frac{i\hat{H}t}{\hbar}} \, |E_i\rangle \\ &= \sum_{i=0}^{N} \langle E_i|\psi(0)|E_i|\psi(0)\rangle \, e^{-\frac{iE_it}{\hbar}} \, |E_i\rangle \, ,\end{aligned}$$

where in the last step, we have used the result obtained in Section 1.2.2. Therefore, as long as we can express a state $|\psi\rangle$ as a superposition of stationary states, its evolution is trivially found.

1.3 Unitary Transformations

Depending on the problem that we intend to solve, it can be beneficial to introduce a transformation to the state vector that simplifies the solution. This is somewhat analogue to the change of coordinates that we sometimes apply to exploit a problem's symmetry.

Let us consider an unitary transformation $\hat{U}$, such that $\hat{U}^\dagger \hat{U} = \mathbb{I}$ (this is important to ensure that the total probability of the transformed state is conserved). We can introduce the state $\left|\tilde{\psi}\right\rangle$ defined as

$$\left|\tilde{\psi}\right\rangle \coloneqq \hat{U} \, |\psi\rangle \, . \tag{1.28}$$

Now, we would like to see how the Hamiltonian transforms after the state transformation $|\psi\rangle \to \left|\tilde{\psi}\right\rangle$.

Lemma 1.17. *For a system defined by* $\left(\hat{H}, |\psi\rangle\right)$ *that undergoes the unitary transformation* $|\psi\rangle \to \hat{U}|\psi\rangle = \left|\tilde{\psi}\right\rangle$*, the Hamiltonian transforms as*

$$\hat{H} \to \hat{U}\hat{H}\hat{U}^\dagger + i\hbar\frac{\partial}{\partial t}\left\{\hat{U}\right\}\hat{U} = \hat{\tilde{H}}.$$

Proof. From equation 1.28 we get $|\psi\rangle = U^\dagger\left|\tilde{\psi}\right\rangle$. Substituting for $|\psi\rangle$ in the Schrödinger equation, we get

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left\{\hat{U}^\dagger\left|\tilde{\psi}\right\rangle\right\} &= \hat{H}\hat{U}^\dagger\left|\tilde{\psi}\right\rangle \\
i\hbar\left(\frac{\partial}{\partial t}\left\{\hat{U}^\dagger\right\}\left|\tilde{\psi}\right\rangle + \hat{U}^\dagger\frac{\partial}{\partial t}\left\{\left|\tilde{\psi}\right\rangle\right\}\right) &= \hat{H}\hat{U}^\dagger\left|\tilde{\psi}\right\rangle \\
i\hbar\hat{U}^\dagger\frac{\partial}{\partial t}\left|\tilde{\psi}\right\rangle &= \hat{H}\hat{U}^\dagger\left|\tilde{\psi}\right\rangle - i\hbar\frac{\partial}{\partial t}\left\{\hat{U}^\dagger\right\}\left|\tilde{\psi}\right\rangle \\
i\hbar\frac{\partial}{\partial t}\left|\tilde{\psi}\right\rangle &= \left(\hat{U}\hat{H}\hat{U}^\dagger - i\hbar\,\hat{U}\frac{\partial}{\partial t}\left\{\hat{U}^\dagger\right\}\right)\left|\tilde{\psi}\right\rangle.
\end{aligned}$$

We observe that we are almost done with the proof and we just need to change the form of the last term in the r.h.s. To this end we consider

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\{\mathbb{I}\} \\
&= \frac{\partial}{\partial t}\left\{\hat{U}\hat{U}^\dagger\right\} \\
&= \frac{\partial}{\partial t}\left\{\hat{U}\right\}\hat{U}^\dagger + \hat{U}\frac{\partial}{\partial t}\left\{\hat{U}^\dagger\right\}.
\end{aligned}$$

Thus, we see that $\hat{U}\frac{\partial}{\partial t}\left\{\hat{U}^\dagger\right\} = -\frac{\partial}{\partial t}\left\{\hat{U}\right\}\hat{U}^\dagger$ and we get that

$$i\hbar\frac{\partial}{\partial t}\left|\tilde{\psi}\right\rangle = \left(\hat{U}\hat{H}\hat{U}^\dagger + i\hbar\,\frac{\partial}{\partial t}\left\{\hat{U}\right\}\hat{U}^\dagger\right)\left|\tilde{\psi}\right\rangle. \tag{1.29}$$

By comparing terms in the equation above we get that

$$\hat{\tilde{H}} = \hat{U}\hat{H}\hat{U}^\dagger + i\hbar\,\frac{\partial}{\partial t}\left\{\hat{U}\right\}\hat{U}^\dagger \tag{1.30}$$

□

1.3.1 Interaction Picture

Of particular relevance is the transformation to the *interaction picture* (often called Dirac picture). This is often done to "remove" the effect of known/trivial dynamics of a system.

Let us consider a system with an Hamiltonian $\hat{H}$ that can be separated as follows

$$\hat{H} = \hat{H}_0 + \hat{H}_1. \tag{1.31}$$

We woulfd like to "hide" the dynamics that arise from $\hat{H}_0$. To this end, we consider the unitary transformation given by

$$U(t) = e^{\frac{i\hat{H}_0 t}{\hbar}}, \tag{1.32}$$

such that the state transforms as

$$|\psi(t)\rangle \to e^{\frac{i\hat{H}_0 t}{\hbar}} |\psi(t)\rangle = |\psi_I(t)\rangle, \tag{1.33}$$

where $|\psi_I(t)\rangle$ is the system state vector in the **interaction picture**. We can now apply equation 1.30 to find the Hamiltonian in the interaction picture as

$$\begin{aligned}
\hat{H} \to \hat{H}_I &= e^{\frac{i\hat{H}_0 t}{\hbar}} \left(\hat{H}_0 + \hat{H}_1\right) e^{-\frac{i\hat{H}_0 t}{\hbar}} + i\hbar \frac{\partial}{\partial t} \left\{ e^{\frac{i\hat{H}_0 t}{\hbar}} \right\} e^{-\frac{i\hat{H}_0 t}{\hbar}} \\
&= \hat{H}_0 + e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{H}_1 e^{-\frac{i\hat{H}_0 t}{\hbar}} - \hat{H}_0 \\
&= e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{H}_1 e^{-\frac{i\hat{H}_0 t}{\hbar}},
\end{aligned}$$

Where we have assumed that $\hat{H}_0$ is time independent. Therefore, as we have shown in the previous section, the state in interaction picture $|\psi_I(t)\rangle$ evolves according to the modified Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi_I(t)\rangle = \hat{H}_I |\psi_I(t)\rangle \tag{1.34}$$

In addition, we want to see how any operator $\hat{A}$ transforms in the interaction picture. We start by computing the expectation for the the operator in the interaction picture $\hat{A}$ as the state $|\psi_I(t)\rangle$ evolves in time

$$\begin{aligned}
\langle \hat{A} \rangle_{\psi_I}(t) &= \left\langle \psi_I(t) \middle| \hat{A} \middle| \psi_I(t) \right\rangle \\
&= \left\langle \psi(t) \middle| e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{A} e^{-\frac{i\hat{H}_0 t}{\hbar}} \middle| \psi(t) \right\rangle \\
&= \left\langle \psi(t) \middle| \hat{A}_I(t) \middle| \psi(t) \right\rangle,
\end{aligned}$$

where we have defined the operator in interaction picture as

$$\hat{A} \to \hat{A}_I(t) \coloneqq e^{\frac{i\hat{H}_0 t}{\hbar}} \hat{A} e^{-\frac{i\hat{H}_0 t}{\hbar}}, \tag{1.35}$$

thus transforming exactly as in the case of the Hamiltonian. Since in the interaction picture, both state vectors and operators are time-dependent, we should find the time evolution equation for the operator $\hat{A}_I(t)$. To this end we write

down

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_I(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{A}e^{-\frac{i\hat{H}_0 t}{\hbar}}\right\} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left\{e^{\frac{i\hat{H}_0 t}{\hbar}}\right\}\hat{A}e^{-\frac{i\hat{H}_0 t}{\hbar}} + e^{\frac{i\hat{H}_0 t}{\hbar}}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\hat{A}\right\}e^{-\frac{i\hat{H}_0 t}{\hbar}} + e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{A}\frac{\mathrm{d}}{\mathrm{d}t}\left\{e^{-\frac{i\hat{H}_0 t}{\hbar}}\right\} \\
&= \frac{i}{\hbar}\hat{H}_0\, e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{A}e^{-\frac{i\hat{H}_0 t}{\hbar}} - \frac{i}{\hbar}e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{A}\hat{H}_0\, e^{-\frac{i\hat{H}_0 t}{\hbar}} \\
&= \frac{1}{i\hbar}e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{A}e^{-\frac{i\hat{H}_0 t}{\hbar}}\hat{H}_0 - \frac{1}{i\hbar}\hat{H}_0\, e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{A}e^{-\frac{i\hat{H}_0 t}{\hbar}} \\
&= \frac{1}{i\hbar}\left[\hat{A}_I(t),\, \hat{H}_0\right],
\end{aligned}$$

where we have assumed that $\hat{A}$ does not explicitly depend on time. This is the familiar Liouville-Von Neumann equation of motion.

1.4 Quantum Harmonic Oscillator

As we will see later on in the manuscript, one particular system tends to appear in many quantum mechanics problems. This system is the so called *quantum harmonic oscillator*. As the name suggests, it is the "quantum extension" to the classic harmonic oscillator. We know from classical physics, that its Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{1.36}$$

If we exchange all the quantities (H, p and x) for their corresponding operators and introduce the canonical commutation relation

$$[\hat{x}, \hat{p}] = i\hbar\mathbb{I}, \tag{1.37}$$

we obtain the Hamiltonian for the quantised harmonic oscillator as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{1.38}$$

We can then rearrange to further simplify (by a clever diagonalization of the Hamiltonian) the expression above. To this end, we introduce the annihilation and creation operators, respectively:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \tag{1.39}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right). \tag{1.40}$$

We can solve for $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. If we go through the algebra we find that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a} + \hat{a}^\dagger\right), \tag{1.41}$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}} \left(\hat{a}^\dagger - \hat{a}\right). \tag{1.42}$$

We can then substitute back in equation 1.36 to get

$$\begin{aligned}
\hat{H} &= \frac{1}{2m}\left[i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right)\right]^2 + \frac{m\omega^2}{2}\left[\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)\right]^2 \\
&= \frac{\hbar\omega}{4}\left[\left(\hat{a} + \hat{a}^\dagger\right)^2 - \left(\hat{a}^\dagger - \hat{a}\right)^2\right] \\
&= \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right).
\end{aligned}$$

To further simplify the expression above, we aim at finding the form of the commutator between ladder operators.

Lemma 1.18. *The commutator between the annihilation and creation (ladder) operators is the identity operator, namely*

$$\left[\hat{a},\, \hat{a}^\dagger\right] = \mathbb{I} \tag{1.43}$$

Proof. We simply plug in equations 1.39 and 1.40, by then applying the canonical commutation relation 1.37 we get

$$\begin{aligned}
\left[\hat{a},\, \hat{a}^\dagger\right] &= \frac{m\omega}{2\hbar}\left[\hat{x} + \frac{i}{m\omega}\hat{p},\, \hat{x} - \frac{i}{m\omega}\hat{p}\right] \\
&= \frac{m\omega}{2\hbar}\left([\hat{x},\, \hat{x}] - \frac{i}{m\omega}[\hat{x},\, \hat{p}] + \frac{i}{m\omega}[\hat{p},\, \hat{x}] + \frac{1}{m^2\omega^2}[\hat{p},\, \hat{p}]\right) \\
&= \frac{1}{2\hbar}\, 2\,[\hat{x},\, \hat{p}] = \mathbb{I}
\end{aligned}$$

□

Thus, we can finalise our diagonalization. In fact, applying the result we have found above we get that

$$\hat{H} = \hbar\omega\left(\frac{1}{2} + \hat{a}^\dagger\hat{a}\right) = \hbar\omega\left(\frac{1}{2} + \hat{n}\right), \tag{1.44}$$

where we have defined the number operator $\hat{n} \coloneqq \hat{a}^\dagger\hat{a}$.

1.4.1 Fock States

We now introduce the *Fock state* $|n\rangle$ as the eigenstate of the number operator $\hat{n}$, namely

$$\hat{n}\,|n\rangle = n\,|n\rangle\,. \tag{1.45}$$

We are interested in computing the action of ladder operators on a Fock state. To this end we write the following

$$\hat{n}\hat{a}\,|n\rangle = (\hat{a}\hat{n} + [\hat{n},\,\hat{a}])\,|n\rangle\,, \tag{1.46}$$

$$\hat{n}\hat{a}^\dagger\,|n\rangle = \left(\hat{a}^\dagger\hat{n} + \left[\hat{n},\,\hat{a}^\dagger\right]\right)|n\rangle\,. \tag{1.47}$$

We continue by computing the commutators of interest, namely

$$[\hat{n},\,\hat{a}] = \left[\hat{a}^\dagger,\,\hat{a}\right]\hat{a} = -\hat{a}, \tag{1.48}$$

$$\left[\hat{n},\,\hat{a}^\dagger\right] = \hat{a}^\dagger\left[\hat{a},\,\hat{a}^\dagger\right] = \hat{a}^\dagger. \tag{1.49}$$

We can now substitute back into equations 1.46 and 1.47 to get

$$\hat{n}\hat{a}\,|n\rangle = \hat{a}\,(\hat{n} - \mathbb{I})\,|n\rangle = (n-1)\,\hat{a}\,|n\rangle\,, \tag{1.50}$$

$$\hat{n}\hat{a}^\dagger\,|n\rangle = \hat{a}^\dagger\,(\hat{n} + \mathbb{I})\,|n\rangle = (n+1)\,\hat{a}^\dagger\,|n\rangle\,. \tag{1.51}$$

The result above tells us that both $\hat{a}\,|n\rangle$ and $\hat{a}^\dagger\,|n\rangle$ are eigenstates of the number operator $\hat{n}$, and thus are also Fock states. The eigenvalues tell us that the annihilation operator $\hat{a}$ removes one "excitation" from the harmonic oscillator, while the creation operator $\hat{a}^\dagger$ adds one. Therefore the action of the ladder operators must be

$$\hat{a}\,|n\rangle = \gamma_n\,|n-1\rangle\,, \tag{1.52}$$

$$\hat{a}^\dagger\,|n\rangle = \tilde{\gamma}_n\,|n+1\rangle\,. \tag{1.53}$$

Here γ_n and $\tilde{\gamma}_n$ are constants and need to be computed. One approach is to compute the expectation value of the number operator and expand. For the annihilation operator we get

$$\begin{aligned} n &= \langle n\,|\,\hat{n}\,|\,n\rangle \\ &= \left\langle n\,\middle|\,\hat{a}^\dagger\hat{a}\,\middle|\,n\right\rangle \\ &= \gamma_n\gamma_n^\star\,\langle n-1\,|\,n-1\rangle = |\gamma_n|^2. \end{aligned}$$

Therefore we get that $\gamma_n = \sqrt{n}$. We can repeat the same process for the creation operator

$$\begin{aligned} n &= \langle n\,|\,\hat{n}\,|\,n\rangle \\ &= \left\langle n\,\middle|\,\hat{a}^\dagger\hat{a}\,\middle|\,n\right\rangle \\ &= \left\langle n\,\middle|\,\hat{a}\hat{a}^\dagger + \left[\hat{a}^\dagger,\,\hat{a}\right]\,\middle|\,n\right\rangle \\ &= \left\langle n\,\middle|\,\hat{a}\hat{a}^\dagger\,\middle|\,n\right\rangle - \left\langle\hat{n}\,\middle|\,\left[\hat{a},\,\hat{a}^\dagger\right]\,\middle|\,n\right\rangle \\ &= \tilde{\gamma}_n\tilde{\gamma}_n^\star\,\langle n+1\,|\,n+1\rangle - 1 = |\tilde{\gamma}_n|^2 - 1. \end{aligned}$$

Finally, we get that $\gamma_n = \sqrt{n+1}$. Therefore, we can write the complete action of the ladder operators as

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{1.54}$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{1.55}$$

1.5 Schrieffer–Wolff Transformation

The *Schrieffer-Wolff* transformation is a unitary transformation used to determine an effective Hamiltonian by decoupling (or tracing out the effect of) weakly interacting subspaces. It is the equivalent of a perturbative correction, but performed by means of operators.

To better explain the utility of the SW transform, let's consider the time-independent Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \tag{1.56}$$

where $\hat{H}_0$ is a Hamiltonian with known eigenstates $\{|m\rangle\}$ - in other words it is diagonal - and $\hat{H}_1$ is a small perturbation. Without loss of generality, we assume $\hat{H}_1$ to be purely off-diagonal in the basis of $\hat{H}_0$, namely $\langle m|\,\hat{H}_1\,|m\rangle = 0\,\forall m$.

We now introduce the unitary transformation as in 1.3, namely of the form

$$\hat{H}' = e^{\hat{S}}\hat{H}e^{-\hat{S}}. \tag{1.57}$$

The expression above can be expanded using *Baker–Campbell–Hausdorff formula* to obtain

$$\hat{H}' = \hat{H} + [\hat{S},\hat{H}] + \frac{1}{2}[\hat{S},[\hat{S},\hat{H}]] + \ldots, \tag{1.58}$$

$$= \hat{H}_0 + \hat{H}_1 + [\hat{S},\hat{H}_0] + [\hat{S},\hat{H}_1] + \frac{1}{2}[\hat{S},[\hat{S},\hat{H}_0]] + \frac{1}{2}[\hat{S},[\hat{S},\hat{H}_1]] + \ldots. \tag{1.59}$$

We now chose $\hat{S}$ such that we diagonalise $\hat{H}'$ to first order in $\hat{H}_1$. This is done by setting

$$[\hat{S},\hat{H}_0] = -\hat{H}_1. \tag{1.60}$$

We note that $\hat{S}$ needs to be off-diagonal. This is because $\hat{H}_0$ and $\hat{H}_1$ are diagonal and off-diagonal by construction. Therefore, the only way to make the commutator off-diagonal - and thus matching the right hand side - is to make $\hat{S}$ also off-diagonal.

Plugging the expression above into equation 1.59 gives the diagonalised expression as

$$\hat{H}' = \hat{H}_0 + [\hat{S},\hat{H}_1] - \frac{1}{2}[\hat{S},\hat{H}_1] + \frac{1}{2}[\hat{S},[\hat{S},\hat{H}_1]] + \ldots \tag{1.61}$$

$$= \hat{H}_0 + \frac{1}{2}[\hat{S},\hat{H}_1] + O(\hat{V}^3) \tag{1.62}$$

We see that in the expression above we have successfully diagonalised $\hat{H}'$ as $[\hat{S},\hat{H}_1]$ must be diagonal up to the first order (as both $\hat{S}$ and $\hat{H}_1$ are off-diagonal).

2 Atom-Field Interaction

We now turn our attention to a a system that is composed of two subsystems, namely

- a set of particles at positions $\mathbf{r}_\alpha$ and having mass m_α. The particles will also have a charge q_α and
- the quantised electromagnetic field.

Therefore, the total Lagrangian for the compounded system is given by the individual Lagrangians for each sub-system, as well as an interaction term. It turns out that the Lagrangian that gives rise to Maxwell-Lorentz equations is

$$\mathcal{L} = \underbrace{\sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 - V_{\text{Coulomb}}}_{\mathcal{L}_{\text{Particle}}} + \mathcal{L}_{\text{Field}} + \underbrace{\int \mathbf{j}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) \, \mathrm{d}^3\mathbf{r}}_{\mathcal{L}_{\text{Interaction}}}, \tag{2.1}$$

where $\mathbf{A}(\mathbf{r})$ is the electromagnetic vector potential and $\mathbf{j}(\mathbf{r})$ is the current density, which for the set of particles is given by

$$\mathbf{j}(\mathbf{r}) = \sum_\alpha q_\alpha \dot{\mathbf{r}}_\alpha \delta(\mathbf{r} - \mathbf{r}_a). \tag{2.2}$$

The expression above can be intuitively seen as the charges moving and thus creating localised currents, hence the delta term. Therefore the full Lagrangian becomes

$$\mathcal{L} = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 - V_{\text{Coulomb}} + \mathcal{L}_{\text{Field}} + \sum_\alpha q_\alpha \int \dot{\mathbf{r}}_\alpha \cdot \mathbf{A}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_a) \, \mathrm{d}^3\mathbf{r} \tag{2.3}$$

$$= \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 - V_{\text{Coulomb}} + \mathcal{L}_{\text{Field}} + \sum_\alpha q_\alpha \dot{\mathbf{r}}_\alpha \cdot \mathbf{A}(\mathbf{r}_\alpha). \tag{2.4}$$

To proceed, we need to identify the conjugate variables of motion. In particular, the conjugate momentum for particle α is given by

$$\begin{aligned} \mathbf{p}_\alpha &= \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_\alpha} \\ &= m_\alpha \dot{\mathbf{r}}_\alpha + q_\alpha \mathbf{A}(\mathbf{r}_\alpha). \end{aligned}$$

We immediately notice that the conjugate momentum for a particle α is no longer given by the kinetic momentum $m_\alpha \dot{\mathbf{r}}_\alpha$. In fact, now the particle "feels" the presence of the field and this is reflected in the modified momentum. This is not ideal and we will get back to this.

From this, we can derive the Hamiltonian for the compounded system as

$$\begin{aligned} H &= \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}^2 + V_{\text{Coulomb}} + H_{\text{Field}} \\ &= \sum_\alpha \frac{1}{2m_\alpha} \left[\mathbf{p}_\alpha - q_\alpha \mathbf{A}(\mathbf{r}_\alpha)\right]^2 + V_{\text{Coulomb}} + H_{\text{Field}} \\ &= \underbrace{\sum_\alpha \frac{\mathbf{p}_\alpha}{2m_\alpha} + V_{\text{Coulomb}}}_{H_{\text{Particle}}} + H_{\text{Field}} \underbrace{- \sum_\alpha \frac{q_\alpha}{m_\alpha} \mathbf{p}_\alpha \cdot \mathbf{A}(\mathbf{r}_\alpha) + \sum_\alpha \frac{q_\alpha^2}{2m_\alpha} \mathbf{A}^2(\mathbf{r}_\alpha)}_{H_{\text{Interaction}}}. \end{aligned}$$

For weak fields, usually only one of the two interaction terms dominates. In particular, in quantum optics, the energy arising from the polarizing effect of the field on the atom is negligible (we are not close to ionising the atom). Therefore, we write

$$H = \sum_\alpha \frac{\mathbf{p}_\alpha}{2m_\alpha} + V_{\text{Coulomb}} + H_{\text{Field}} - \sum_\alpha \frac{q_\alpha}{m_\alpha} \mathbf{p}_\alpha \cdot \mathbf{A}(\mathbf{r}_\alpha) \tag{2.5}$$

2.1 Göppert-Mayer Gauge Transformation

As we said before, the conjugate momentum for a particle α is dependent on the field. To recover an Hamiltonian that can be split in the "particle", "field" and "interaction" components we apply the so called *Göppert-Mayer gauge transformation*. For the transformation to be valid we need to assume three things:

1. the system of charges is **globally neutral**, namely
$$\sum_\alpha q_\alpha = 0; \tag{2.6}$$
2. the field is **non-dynamical**, or in other words it is not affected by the particle's presence;
3. all the charges are close to the origin, this is the so called **long wavelength approximation**, that states
$$\max_\alpha \{\mathbf{r}_\alpha\} \ll \lambda, \tag{2.7}$$
where λ is the wavelength associated with the field.

The transformation relies on the fact that the Lagrangian is invariant under the transformation given by

$$\mathcal{L} \to \mathcal{L} + \frac{\mathrm{d}}{\mathrm{d}t} F(\mathbf{r}, t). \tag{2.8}$$

We now let $F = -\sum_\alpha q_\alpha \mathbf{r}_\alpha \cdot \mathbf{A}(\mathbf{r}_\alpha)$ and substitute back in equation 2.4 to get

$$\mathcal{L} = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 - V_{\text{Coulomb}} + \mathcal{L}_{\text{Field}} + \sum_\alpha q_\alpha \dot{\mathbf{r}}_\alpha \cdot \mathbf{A}(0,t) - \frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha q_\alpha \mathbf{r}_\alpha \cdot \mathbf{A}(0,t), \tag{2.9}$$

where we have substituted for $\mathbf{A}(\mathbf{r}_\alpha) \to \mathbf{A}(0,t)$ by making use of the long wavelength approximation. After simplifying the expression above we get

$$\mathcal{L} = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 - V_{\text{Coulomb}} + \mathcal{L}_{\text{Field}} - \sum_\alpha q_\alpha \mathbf{r}_\alpha \cdot \dot{\mathbf{A}}(0,t). \tag{2.10}$$

Finally we notice that the conjugate momentum $\mathbf{p}_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_\alpha}$ coincides with the kinetic momentum $m_\alpha \dot{\mathbf{r}}_\alpha$. We introduce the dipole operator

$$\mathbf{d} = \sum_\alpha q_\alpha \mathbf{r}_\alpha, \tag{2.11}$$

that intuitively measures the displacement of each charge giving rise to a dipole. In addition, we know that the electric field is given by

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}. \tag{2.12}$$

Therefore the Hamitonian associated to the Lagrangian in 2.10 becomes the following

$$\begin{aligned} H &= \sum_\alpha \mathbf{p}_\alpha \dot{\mathbf{r}}_\alpha - \mathcal{L} \\ &= \sum_\alpha (m_\alpha \dot{\mathbf{r}}_\alpha)\dot{\mathbf{r}}_\alpha - \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 + V_{\text{Coulomb}} + \mathcal{L}_{\text{Field}} - \mathbf{d} \cdot \mathbf{E}(0,t) \\ &= \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 + V_{\text{Coulomb}} + \mathcal{L}_{\text{Field}} - \mathbf{d} \cdot \mathbf{E}(0,t) \end{aligned}$$

Since we consider the field to be non-dynamical, we can ignore the $\mathcal{L}_{\text{Field}}$ term and we get to the usual dipole interaction hamiltonian

$$H = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 + V_{\text{Coulomb}} - \mathbf{d} \cdot \mathbf{E}(0,t). \tag{2.13}$$

2.2 PZW Transform

While the Göppert-Mayer gauge transform allowed us to recover the kinematic form of the particles' conjugate momenta, we were able to do so by considering the electric field to be non-dynamical. This is a very restrictive limitation and we should investigate weather the dipole Hamiltonian in 2.13 is also valid for dynamical fields.

Work In Progress

2.3 Jaynes-Cummings Hamiltonian

We will now compute the Hamiltonian for a single two-level emitter (i.e. a simplified model of an atom, or a *qubit*) coupled to a single electromagnetic field cavity mode. Here are the main assumptions for the Jaynes-Cummings model to be valid:

- the emitter only has two energy levels and they are separated by $\hbar\omega_{eg}$;
- the emitter couples to a single cavity mode represented by the operator $\hat{a}_c$ and a single polarization σ_c;
- the **rotating wave approximation** is valid. We will delve in more detail later in this section.

We start by defining the two levels of the atom $|g\rangle$ and $|e\rangle$. We define the ground state energy $E_g = 0$ and the excited state energy $E_e = \hbar\omega_{eg}$. Therefore, the emitter Hamiltonian immediately becomes

$$\hat{H}_e = \hbar\omega_{eg} |e\rangle \langle e| = \hbar\omega_{eg}\hat{\sigma}_{ee}, \tag{2.14}$$

where we have defined the excited state projector $\hat{\sigma}_{ee} \coloneqq |e\rangle \langle e|$. In addition, we know that the Hamiltonian of the cavity is given by

$$\hat{H}_c = \sum_{k,\boldsymbol{\sigma}_k} \hbar\omega_k \left(\hat{a}^\dagger_{k\boldsymbol{\sigma}_k} \hat{a}_{k\boldsymbol{\sigma}_k} + \frac{1}{2} \right). \tag{2.15}$$

By construction, we only consider one mode and one polarization. Therefore, by ignoring the constant term $1/2$ the Hamiltonian reduces to

$$\hat{H}_c = \hbar\omega_c \hat{a}^\dagger_c \hat{a}_c. \tag{2.16}$$

Plugging in terms into the dipole Hamiltonian in 2.13, we get the following expression

$$\hat{H} = \hbar\omega_{eg}\hat{\sigma}_{ee} + \hbar\omega_c \hat{a}^\dagger_c \hat{a}_c - \hat{\mathbf{d}} \cdot \hat{\mathbf{E}}. \tag{2.17}$$

We now need to expand the interaction (last) term in the expression above. We begin by recalling the general form of the electric field operator

$$\hat{\mathbf{E}} = \sum_{k,\boldsymbol{\sigma}_k} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \sigma_k \left(\hat{a}_{k\sigma_k} + \hat{a}^\dagger_{k\sigma_k} \right), \tag{2.18}$$

where V is the cavity mode volume and ε_0 is the vacuum permittivity constant. From here, since we consider only one cavity mode we can simplify the expression above down to

$$\hat{\mathbf{E}} = \sqrt{\frac{\hbar\omega_c}{2\varepsilon_0 V}} \, \boldsymbol{\sigma}_c \left(\hat{a}_c + \hat{a}^\dagger_c \right). \tag{2.19}$$

Furthermore let us expand the dipole operator $\hat{\mathbf{d}}$ by leveraging the *completeness relationship* as follows

$$\begin{aligned} \hat{\mathbf{d}} &= \sum_i |i\rangle \langle i| \, \hat{\mathbf{d}} \sum_j |j\rangle \langle j| \\ &= \sum_{i,j} |i\rangle \langle i| \, \hat{\mathbf{d}} \, |j\rangle \langle j| \\ &= \sum_{i,j} \boldsymbol{\mu}_{ij} |i\rangle \langle j|, \end{aligned}$$

where $|k\rangle$ are the energy eigenstates of the emitter. Since we are considering a two-levels system coupling to a single mode (and therefore polarization), the equation above becomes

$$\hat{\mathbf{d}} = \boldsymbol{\mu}_{eg}(|e\rangle\langle g| + |g\rangle\langle e|) = \boldsymbol{\mu}_{eg}(\hat{\sigma}_{eg} + \hat{\sigma}_{ge}), \tag{2.20}$$

where we have introduced the raising and lowering operators $\hat{\sigma}_{eg} \coloneqq |e\rangle\langle g|$ and $\hat{\sigma}_{ge} \coloneqq |g\rangle\langle e|$ respectively. In addition, we have also assumed $\boldsymbol{\mu}_{eg} = \boldsymbol{\mu}_{ge}$ and that $\boldsymbol{\mu}_{ii} = 0\,\forall i$. From this point we substitute back into equation 2.17 to get

$$\hat{H} = \hbar\omega_{eg}\hat{\sigma}_{ee} + \hbar\omega_c\hat{a}_c^\dagger\hat{a}_c - \sqrt{\frac{\hbar\omega_c}{2\varepsilon_0 V}}\,(\boldsymbol{\mu}_{eg}\cdot\boldsymbol{\sigma}_c)\left(\hat{a}_c + \hat{a}_c^\dagger\right)\otimes(\hat{\sigma}_{eg} + \hat{\sigma}_{ge}) \tag{2.21}$$

$$= \hbar\omega_{eg}\hat{\sigma}_{ee} + \hbar\omega_c\hat{a}_c^\dagger\hat{a}_c - \hbar g_c\left(\hat{a}_c + \hat{a}_c^\dagger\right)\otimes(\hat{\sigma}_{eg} + \hat{\sigma}_{ge}), \tag{2.22}$$

where we have introduced the coupling rate g_c that characterises the interaction strength between the cavity mode and the emitter and is defined as

$$g_c = \sqrt{\frac{\omega_c}{2\hbar\varepsilon_0 V}}\,(\boldsymbol{\mu}_{eg}\cdot\boldsymbol{\sigma}_c). \tag{2.23}$$

Before we proceed any further, we will switch to *interaction picture* by choosing our unitary as

$$\hat{U} = e^{\frac{i}{\hbar}\left(\hbar\omega_{eg}\hat{\sigma}_{ee} + \hbar\omega_c\hat{a}_c^\dagger\hat{a}_c\right)t} = e^{i\omega_{eg}\hat{\sigma}_{ee}t}\otimes e^{i\omega_c\hat{\sigma}_{ee}\hat{a}_c^\dagger\hat{a}_c}, \tag{2.24}$$

where we have leveraged the fact that operators belonging to different Hilbert spaces always commute. Precisely following what we have done in Section 1.3.1, we obtain the following hamiltonian

$$\hat{H}_I = \hbar g_c\left[\hat{a}_c\hat{\sigma}_{eg}e^{i(-\omega_c+\omega_{eg})t} + \hat{a}_c^\dagger\hat{\sigma}_{eg}e^{i(+\omega_c+\omega_{eg})t} + \text{h.c.}\right]. \tag{2.25}$$

We can apply the so-called *Rotating Wave Approximation*. The approximation is valid for

- **weak coupling regime**: $g_c \ll \omega_{eg} + \omega_c$;
- **resonant condition**: $|\omega_{eg} - \omega_c| \ll \omega_{eg}$.

If these conditions are satisfied, we can write the Jaynes-Cummings Hamiltonian as

$$H_{JCM} = \hbar g_c\left[\hat{a}_c\sigma_{eg}e^{i(\omega_{eg}-\omega_c)} + \hat{a}_c^\dagger\sigma_{ge}e^{-i(\omega_{eg}-\omega_c)}\right]. \tag{2.26}$$

2.4 Spontaneous emission

System to be considered Here, we consider the case of a single emitter interactinve with a continuum of radiation field modes in free space. We assume that the emitter is initially in the excited state $|e\rangle$, more specifically, the statevector of the system is given by:

$$|\psi(0)\rangle = |e, \underline{0}\rangle \tag{2.27}$$

where $|\underline{0}\rangle$ is the vacuum state of the radiation field. Due to electric dipole coupling in the RWA limit, $e, \underline{0}$ will couple to states, resulting in

$$\hat{\sigma}_{ge}\hat{a}_k \, |e, \underline{0}\rangle = |g, 1_k\rangle \tag{2.28}$$

where $|1_k\rangle$ denote a state with one-photon in mode k.

In this system, we assume an Ansatz, (following Wigener Weisskopf):

$$|\psi(t)\rangle = a_{e,0}(t) \, |e, \underline{0}\rangle + \sum_k a_{g,k}(t) \, |g, 1_k\rangle \tag{2.29}$$

and solve the Schrödinger equation in the interaction picture to obtain coupled amplitude equations.

With the following Hamiltonian in the interaction picture:

$$\hat{H} = \sum_k i\hbar\Big(g_k \hat{\sigma}_{ge}\hat{a}_k e^{i(\omega_{eg}-\omega_k)t} - g_k^* \hat{a}_k^\dagger \hat{\sigma}_{eg} e^{-i(\omega_{eg}-\omega_k)t}\Big) \tag{2.30}$$

Beginning of calculation The Schrödinger equation in the interaction picture is given by:

$$\frac{\mathrm{d}a_{e,0}(t)}{\mathrm{d}t} = \sum_k g_k a_{g,k}(t) e^{i(\omega_{eg}-\omega_k)t} \tag{2.31}$$

$$\frac{\mathrm{d}a_{g,1_k}(t)}{\mathrm{d}t} = -\sum_k g_k^* a_{e,0}(t) e^{-i(\omega_{eg}-\omega_k)t} \tag{2.32}$$

integrating (2.32) over time and substituting in (2.31) gives:

$$\frac{\mathrm{d}a_{e,0}(t)}{\mathrm{d}t} = -\sum_k |g_k|^2 \int_0^t a_{e,0}(t') e^{i(\omega_{eg}-\omega_k)(t-t')} \, dt' \tag{2.33}$$

Calculation and calculation and calculation Now I will just skip

This, according to the lecture note4, ends up with the following expression for the amplitude $a_{e,0}(t)$:

$$a_{e,0}(t) = -i\frac{\mu_{eg}^2}{6\pi^2\varepsilon_0\hbar c^3} a_{e,0}(t) \int_0^t dt' \int_0^\infty d\omega \frac{\omega^3}{\omega_{eg}-\omega+i\epsilon} \tag{2.34}$$

$$= -\frac{\Gamma}{2} a_{e,0}(t) + i\Delta_{LS} a_{e,0}(t) \tag{2.35}$$

where

$$\Gamma = \frac{\mu_{eg}^2 \omega_{eg}^3}{3\pi^2\varepsilon_0\hbar c^3} \tag{2.36}$$

$$\Delta_{LS} = -\frac{\mu_{eg}^2 \omega_{eg}^3}{6\pi^2\varepsilon_0\hbar c^3} \times P\left[\frac{\omega_{eg}^3}{\omega_{eg}-\omega_k}\right] \tag{2.37}$$

$$\lim_{\epsilon\to 0} \int_0^\infty d\omega \frac{\omega^3}{\omega_{eg}-\omega+i\epsilon} = P\left[\frac{\omega_{eg}^3}{\omega_{eg}-\omega}\right] - i\pi\delta(\omega_{eg}-\omega) \tag{2.38}$$

Observations

1. We started out with unitary evolution, but ended up with exponential decay of atomic population; while the rate Γ is correct, we are not consistent since we cannot treat the emitter with a wavefunction that discards the field degrees of freedom.

2. Even though only the modes with $\omega_{eg} = \omega$ appears to be responsible for spontaneous emission (delta function in (2.34) and(2.38)), it is the presence of a large bandwidth of modes that ensures the validity of Morkov approximation. If the $d\omega$ integral during the process were restricted to $(\omega_{eg} - \Gamma, \omega_{eg} + \Gamma)$, due to a photonic bandgap, we would not obtain exponential decay.

Motivation for density operator In the example we just considered, wthe emitter becomes correlated with a reservoir / bath (the continuum of radiation field modes in free space), which makes it impossible to describe the emitter with a wavefunction. The reason why we cannot use a wavefunction to represent the emitter is because our information is no longer complete. Therefore, we need a density operator $\hat{\rho}$ to describe the system.

Density operator $\hat{\rho}$ provides the most general representation of our knowledge about a quantum system.

Pure state $\hat{\rho} = |\psi\rangle \langle\psi|$. we have complete information about the sytem. $\hat{\rho}$ is a projector operator into system state vector. $\rho^2 = \rho$.

Mixed state $\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle\psi_i|$. We have incomplete information about the system; we can only say that the system wave-function is $|\psi_i\rangle$ with probability $p_i < 1$. $\hat{\rho}^2 \neq \hat{\rho}$.

2.5 Density operator

Definition of reduced density operator Consider a system A that interacts with another system B; the reduced density operator of the system A is defined as:

$$\hat{\rho}_A = \mathrm{Tr}_B(\hat{\rho}_{AB}) \tag{2.39}$$

where $\hat{\rho_{AB}}$ is the density operator of the joint system.

Properties of the density operator Mathematically, the density operator is a non-negative, Hermitian operator with unit trace.

$$\langle\hat{o}\rangle = \mathrm{Tr}\left(\hat{o}(t)\hat{\rho}\right) = \mathrm{Tr}\left(\hat{o}\hat{\rho}(t)\right) \tag{2.40}$$

$$S = -k_B \,\mathrm{Tr}\left(\hat{\rho}\ln\hat{\rho}\right) \tag{2.41}$$

and, $S = 0$ for a pure state / $S > 0$ for a mixed state.

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = -\frac{i}{\hbar}\Big[\hat{H}, \hat{\rho}(t)\Big] \tag{2.42}$$

thermal state of a single field mode I might add this later.

2.6 Master equation

Goal To expand our derivation of spontaneous emission and to obtain an equation that describes the time evolution of the (reduced) system density operator $\hat{\rho}_S$ in the presence of coherent and dissipative processes.

System to be considered We only consider an interaction Hamiltonian coupling the emitter (system) to the radiation field (reservoir). The Hamiltonian is given by:

$$\hat{H}_{int} = \sum_i \beta_i \underbrace{\hat{A}_S^{(i)}}_{\text{system operator}} \underbrace{\hat{B}_R^{(i)}}_{\text{reservoir operator}} \tag{2.43}$$

$$= \sum_k i\hbar g_k \left(\underbrace{\hat{\sigma}_{eg}}_{S} \underbrace{\hat{a}_k}_{R} - \underbrace{\hat{a}_k^\dagger}_{R} \underbrace{\hat{\sigma}_{ge}}_{S} \right) \tag{2.44}$$

beginning of calculation We start from Liouville-von Neumann equationl integrate and substitute back on the RHS and then take a trace over the reservoir. We do this calculation in the interaction picture.

The step-by-step calculation is given here. First, we start with the Liouville-von Neumann equation in the interaction picture:

$$\frac{\mathrm{d}\hat{\rho}_{SR}(t)}{\mathrm{d}t} = -\frac{i}{\hbar}\Big[\hat{H}_{int}(t), \hat{\rho}_{SR}(t)\Big] \tag{2.45}$$

$$= -\frac{1}{\hbar^2}\int_0^t dt' \, \Big[\hat{H}_{int}(t), \Big[\hat{H}_{int}(t'), \hat{\rho}_{SR}(t')\Big]\Big] \tag{2.46}$$

$$+ \frac{1}{\hbar^2}\int_0^t dt' \Big[\hat{H}_{int}(t), \rho_{SR}(0)\Big] \tag{2.47}$$

The next step is taking the trace over the reservoir:

$$\mathrm{Tr}_R\left(\frac{\mathrm{d}\hat{\rho}_{SR}(t)}{\mathrm{d}t}\right) = \frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} = -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{Tr}_R\left(\left[\hat{H}_{int}(t),\left[\hat{H}_{int}(t'),\hat{\rho}_{SR}(t')\right]\right]\right) \tag{2.48}$$

$= 0$ assuming a thermal reservoir state

$$+\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{Tr}_R\left(\left[\hat{H}_{int}(t),\hat{\rho}_{SR}(0)\right]\right) \tag{2.49}$$

$$= -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{Tr}_R\left(\left[\hat{H}_{int}(t),\left[\hat{H}_{int}(t'),\hat{\rho}_{SR}(t')\right]\right]\right) \tag{2.50}$$

So far, no approximations have been made except for the assumption that the reservoir is in a thermal state.

More calculation To proceed, we make 3 key assumptions to simplify $\hat{\rho}_{SR}(t')$ term in the RHS of the equation above:

1. Born approximation: Interaction are weak
2. Reservoir approximation: reservoir states does not change due to the interaction with the system.
3. Markov approximation: short memory

Mathematically,

$$\hat{\rho}_{SR}(t') \approx \hat{\rho}_S(t')\otimes\hat{\rho}_R(t') + \hat{\rho}_{SR,corr}(t') \qquad (exact) \tag{2.51}$$

$$\approx \hat{\rho}_S(t')\otimes\hat{\rho}_R(t') \qquad \text{(Born approximation)} \tag{2.52}$$

$$\approx \hat{\rho}_S(t')\otimes\hat{\rho}_R \qquad \text{(Reservoir approximation)} \tag{2.53}$$

$$\approx \hat{\rho}_S(t)\otimes\hat{\rho}_R \qquad \text{(Markov approximation)} \tag{2.54}$$

Using this approximation, we can simplify the equation for $\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t}$ to:

$$\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} = -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{Tr}_R\left(\left[\hat{H}_{\mathrm{int}}(t),\left[\hat{H}_{\mathrm{int}}(t'),\hat{\rho}_{SR}\right]\right]\right) \tag{2.55}$$

$$= -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{Tr}_R\left(\left[\hat{H}_{\mathrm{int}}(t),\left[\hat{H}_{\mathrm{int}}(t'),\hat{\rho}_S(t)\hat{\rho}_R\right]\right]\right) \tag{2.56}$$

$$= -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{Tr}_R\Big[\tag{2.57}$$

$$+\underbrace{\hat{H}_{\mathrm{int}}(t)\hat{H}_{\mathrm{int}}(t')\hat{\rho}_S(t)\hat{\rho}_R}_{(1)} \tag{2.58}$$

$$-\underbrace{\hat{H}_{\mathrm{int}}(t)\hat{\rho}_S(t)\hat{\rho}_R\hat{H}_{\mathrm{int}}(t')}_{(2)} \tag{2.59}$$

$$-\underbrace{\hat{H}_{\mathrm{int}}(t')\hat{\rho}_S(t)\hat{\rho}_R\hat{H}_{\mathrm{int}}(t)}_{(3)} \tag{2.60}$$

$$+\underbrace{\hat{\rho}_S(t)\hat{\rho}_R\hat{H}_{\mathrm{int}}(t')\hat{H}_{\mathrm{int}}(t)}_{(4)} \tag{2.61}$$

$$\Big]. \tag{2.62}$$

Assumptions Now we focus on atom-field coupling at $T_{\mathrm{reservoir}} = 0$. Among the terms in the equation above, the non-vanishing terms are only in the form of $\mathrm{Tr}_R\left(\hat{\rho}_R\hat{a}_k\hat{a}_k^\dagger\right)$ and $k \neq k'$. Continuing from the equation above, we have:

$$(2.58) = \sum_k |g_k|^2 \underbrace{\hat{\sigma}_{eg}\hat{\sigma}_{ge}}_{\hat{\sigma}_{ee}}\hat{\rho}_S(t)\exp(i(\omega_{eg}-\omega_k)(t-t')) \tag{2.63}$$

$$(2.59) = \sum_k |g_k|^2 \hat{\sigma}_{ge}\hat{\rho}_S(t)\hat{\sigma}_{eg}\exp(-i(\omega_{eg}-\omega_k)(t-t')) \tag{2.64}$$

$$(2.60) = \sum_k |g_k|^2 \hat{\sigma}_{ge}\hat{\rho}_S(t)\hat{\sigma}_{eg}\exp(i(\omega_{eg}-\omega_k)(t-t')) \tag{2.65}$$

$$(2.61) = \sum_k |g_k|^2 \hat{\rho}_S(t)\underbrace{\hat{\sigma}_{eg}\hat{\sigma}_{ge}}_{\hat{\sigma}_{ee}}\exp(-i(\omega_{eg}-\omega_k)(t-t')). \tag{2.66}$$

Collecting the nonvanishing contributions and using a constant Γ, effective decay rate, the master equation for the reduced density operator becomes

$$\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} = \frac{\Gamma}{2}\{2\hat{\sigma}_{ge}\hat{\rho}_S(t)\hat{\sigma}_{eg} - \hat{\sigma}_{ee}\hat{\rho}_S(t) - \hat{\rho}_S(t)\hat{\sigma}_{ee}\} \tag{2.67}$$

2.7 Optical Bloch equations

3 Electric Field Statistics

In this section we will concentrate on the coherent properties of light and how they can be gauged by the means of field statistics. We will see that we are mainly interested in two different measurements: *electric field correlation function* and *time correlation*.

3.1 Electric Field Correlation Function

The general first order coherence function is given by the expression

$$G^{(1)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \left\langle E^{(-)}(\mathbf{r}_1, t_1) E^{(+)}(\mathbf{r}_2, t_2) \right\rangle. \tag{3.1}$$

For the special case where the electric field correlation is **time-invariant** (i.e. it does not depend on t_1 but rather only on the difference $t_2 - t_1$) and is comparing fields at the **same position** (what can be measure with a *Mach-Zehender interferometer*), the expression reduces to

$$G^{(1)}(\tau) = \left\langle E^{(-)}(t+\tau) E^{(+)}(t) \right\rangle. \tag{3.2}$$

This function can be used to calculate the spectrum of light emitted from a quantum emitter and, most importantly, it can be used to distinguish *coherent* and *incoherent* scattering. It is important to note, that we often use the normalised coherence function, that is indicated with the lower case

$$g^{(1)}(\tau) = \frac{\left\langle E^{(-)}(t+\tau) E^{(+)}(t) \right\rangle}{\sqrt{\left\langle E^{(-)}(t+\tau) E^{(+)}(t+\tau) \right\rangle \left\langle E^{(-)}(t) E^{(+)}(t) \right\rangle}}. \tag{3.3}$$

To find an analytical expression for $G^{(1)}$ we need to compute the time evolution of the electric field emitted by the quantum emitter, namely

$$E^{(+)}(\mathbf{r}, t) = E_0(\mathbf{r}, t) \hat{\sigma}_{ge} \left(t - \frac{|\mathbf{r} - \mathbf{R}|}{c} \right). \tag{3.4}$$

It seems like that *Optical Bloch Equations* might be a starting point. Let us state them here in matrix form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(t) = \begin{bmatrix} -\left(\frac{\gamma}{2} - i\Delta\right) & i\frac{\Omega_L}{2} & 0 \\ i\Omega_L & -\Gamma & -i\Omega_L \\ 0 & -i\frac{\Omega_L}{2} & -\left(\frac{\gamma}{2} - i\Delta\right) \end{bmatrix} \mathbf{u}(t) - \begin{bmatrix} 0 \\ \Gamma \\ 0 \end{bmatrix}, \tag{3.5}$$

with the state vector defined as

$$\mathbf{u}(t) = \begin{bmatrix} \langle \hat{\sigma}_{eg}(t) \rangle \\ \langle \hat{\sigma}_{ee}(t) - \hat{\sigma}_{gg}(t) \rangle \\ \langle \hat{\sigma}_{ge}(t) \rangle \end{bmatrix}. \tag{3.6}$$

The problem here is that we need an expression involving the expectation value of the product $\hat{sigma}_{eg}$ evaluated at different times. To this end, we resort to the *Quantum Regression Theorem.*

Theorem 3.1 (Quantum Regression Theorem)**.** *Let us consider a set of time-varying operators* $\{\hat{x}_i\}$*. If, the expectation value of the time varying operators satisfy the relationship*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \hat{x}_i(t)\right\rangle = M_{ij}\left\langle \hat{x}_j(t)\right\rangle + \lambda_i, \tag{3.7}$$

then the following relationship is also valid

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \hat{x}_i(t+\tau)\hat{x}_k(t)\right\rangle = M_{ij}\left\langle \hat{x}_j(t+\tau)\hat{x}_k(t)\right\rangle + \lambda_i\left\langle \hat{x}_k(t)\right\rangle. \tag{3.8}$$

The proof is rather involved and left out in this script. By comparing terms in equations 3.5 and 3.8 we immediately see that the *Quantum Regression Theorem* can be used to solve for the electric field correlation function. We will now proceed to solve for $G^{(1)}$ for different regimes.

We begin by evaluating the different time constants that appear in the solution of the Optical Bloch Equations. To this end, we calculate the eigenvalues of the state matrix from 3.5, namely we need to solve for s in the eigenvalue equation below

$$\left| s\mathbb{I} - \begin{bmatrix} -\frac{\Gamma}{2} & i\frac{\Omega_L}{2} & 0 \\ i\Omega_L & -\Gamma & -i\Omega_L \\ 0 & -i\frac{\Omega_L}{2} & -\frac{\Gamma}{2} \end{bmatrix}\right| = \begin{vmatrix} s+\frac{\Gamma}{2} & -i\frac{\Omega_L}{2} & 0 \\ -i\Omega_L & s+\Gamma & i\Omega_L \\ 0 & +i\frac{\Omega_L}{2} & s+\frac{\Gamma}{2} \end{vmatrix}, \tag{3.9}$$

$$= (s-s_1)(s-s_2)(s-s_3). \tag{3.10}$$

With the respective solution for the dynamics s_i, $i = 1, 2, 3$ give by

$$s_1 = -\frac{\Gamma}{2}, \tag{3.11}$$

$$s_2 = \frac{3\Gamma}{4} + \sqrt{\frac{\Gamma^2}{16} - \Omega_L^2}, \tag{3.12}$$

$$s_3 = \frac{3\Gamma}{4} - \sqrt{\frac{\Gamma^2}{16} - \Omega_L^2}. \tag{3.13}$$

Therefore, the general solution for the OBE term is given by the following ansatz

$$\left\langle \hat{\sigma}_{eg}(t)\right\rangle = fe^{s_1 t} + ge^{s_2 t} + he^{s_3 t} + we^0, \tag{3.14}$$

where s_i are the solutions of the eigenvalue equation above and the e^0 is the steady-state term that follows from the constant input to the system. We can now apply the *Quantum Regression Theorem* to obtain the expression for the first order coherence function

$$\left\langle \hat{\sigma}_{eg}(t+\tau)\hat{\sigma}_{ge}(t)\right\rangle = \tilde{f}e^{-\frac{\Gamma}{2}\tau} + \tilde{g}e^{\left(\frac{3\Gamma}{4}+\kappa\right)\tau} + e^{\left(\frac{3\Gamma}{4}-\kappa\right)\tau} + \tilde{w}, \tag{3.15}$$

where we have made use of the fact that the same dynamics will appear in the coherent function, since the state matrix is the same and we have defined

$$\kappa = \sqrt{\frac{\Gamma^2}{16} - \Omega_L^2}. \tag{3.16}$$

Note that 3.15 does not show an explicit dependence on time as we assume a time-invariant system. In addition, we can switch to *Heisenberg picture* to incorporate the fast dynamics into the evolution of the coherence function, namely

$$\langle \hat{\sigma}_{eg}(t+\tau)\hat{\sigma}_{ge}(t)\rangle_H = \left[\tilde{f}e^{-\frac{\Gamma}{2}\tau} + \tilde{g}e^{\left(\frac{3\Gamma}{4}+\kappa\right)\tau} + e^{\left(\frac{3\Gamma}{4}-\kappa\right)\tau} + \tilde{w}\right] e^{-i\omega_L\tau}. \tag{3.17}$$

3.1.1 Coherence Function Solution: Weak Drive ($\Omega_L \ll \Gamma$)

In this regime, the

4 Generation of Fields

5 Cavity Optomechanics

In this section we study the interaction between a quantized EM radiation field and a quantum mechanical oscillator.

5.1 Cavity Optomechanics Hamiltonian

Consider a cavity with one mirror fixed and the other mounted on a spring with a displacement x and an equilibrium cavity length l.

A photon in the cavity has an energy ϵ_γ and momentum p_γ given by

$$\varepsilon_\gamma = \hbar\omega_c \qquad\qquad p_\gamma = \hbar k_c = \hbar\frac{\omega_c}{c} \tag{5.1}$$

Upon reflection at the cavity mirror the momentum of the photon is reversed giving a change in momentum

$$\Delta p = 2\hbar k_c = s\hbar\frac{\omega_c}{c} \tag{5.2}$$

The cavity has a round trip time T given by

$$T = 2\frac{l}{c} \tag{5.3}$$

As the photon travels around the cavity it exerts a mean force on the cavity mirrors of

$$\langle F\rangle = \frac{\Delta p}{T} = \hbar\frac{\omega_c}{l} \tag{5.4}$$

This force compresses the spring resulting in potential energy being transferred from a single photon in the cavity

$$U = \langle F \rangle \, x = \hbar \frac{\omega_{\mathrm{c}}}{l} x \tag{5.5}$$

Combining the energy of the radiation field inside the cavity with the potential energy from displacing the spring by each photon in the cavity we arrive at the cavity optomechanics Hamiltonian

$$\hat{H} = \hbar \omega_{\mathrm{c}} \hat{a}^\dagger \hat{a} - \hbar \frac{\omega_{\mathrm{c}}}{l} x \hat{a}^\dagger \hat{a} \tag{5.6}$$

Note so far we have treated the spring as a purely classical mechanical oscillator.

6 Hong-Ou-Mandel Interference

We now consider a modified Hanbury-Brown-Twiss experiment. Instead of inputting light from a single port of the beamsplitter, we input photon wavepackets from both ports. The aim is to generate two single-photon wavepackets and make them interfere at the beamsplitter. To generate the single photons, we rely on a single two-level atom in a cavity. The atom is excited by a pulsed laser. The effect of the cavity is to define a preferred emission direction, while enhancing the photon generation rate through the *Purcell Effect*.

Let us introduce write the photon pulse wavefunction as

$$|\phi(t)\rangle = \hat{\Psi}^\dagger \, |0\rangle \tag{6.1}$$

$$= \int \mathrm{d}k \, e^{ikt} \, \phi^\star(k) \, \hat{a}_k \, |0\rangle \, . \tag{6.2}$$

We note that the operator $\hat{\Psi}^\dagger$ creates a photon wavepacket. This is easily seen if we consider the Fourier relation between position and momentum. Since we are dealing with photon **packets** and the photon is localised in space, it must be that its state contains multiple momentum components.

We now consider two input packets into the HBT setup. The full input state is written as

$$|\phi_a\rangle \, |\phi_b\rangle = \iint \mathrm{d}k_a \, \mathrm{d}k_b \, e^{i(k_a + k_b)t} \, \phi_a^\star(k_a) \, \phi_b^\star(k_b) \, \hat{a}_{k_a} \, \hat{b}_{k_b} \, |0\rangle \, |0\rangle \, . \tag{6.3}$$

We now consider the beamsplitter relationships between input and output modes, namely

$$\hat{a}^\dagger = \frac{\hat{c}^\dagger + \hat{d}^\dagger}{\sqrt{2}}, \tag{6.4}$$

$$\hat{b}^\dagger = \frac{\hat{c}^\dagger - \hat{d}^\dagger}{\sqrt{2}}. \tag{6.5}$$

Therefore, the system state expressed in terms of output modes becomes

$$|\psi\rangle = \frac{1}{2}\iint \mathrm{d}k_a\,\mathrm{d}k_b\,e^{i(k_a+k_b)t}\,\phi_a^\star(k_a)\,\phi_b^\star(k_b)\,\left(\hat{c}_{k_a}^\dagger + \hat{d}_{k_a}^\dagger\right)\left(\hat{c}_{k_b}^\dagger - \hat{d}_{k_b}^\dagger\right)|0\rangle\,|0\rangle\,, \tag{6.6}$$

$$= \iint \mathrm{d}k_a\,\mathrm{d}k_b\,e^{i(k_a+k_b)t}\,\left[\phi_a^\star(k_a)\,\phi_b^\star(k_b) - \phi_a^\star(k_b)\,\phi_b^\star(k_a)\right]\,\hat{c}_{k_a}^\dagger\,\hat{d}_{k_b}^\dagger \tag{6.7}$$

$$+ \frac{1}{2}\int \mathrm{d}k_a\,e^{ik_at}\,\phi_a^\star(k_a)\,\hat{c}_{k_a}^\dagger \int \mathrm{d}k_b\,e^{ik_at}\,\phi_a^\star(k_b)\,\hat{c}_{k_b}^\dagger\,|0\rangle\,|0\rangle \tag{6.8}$$

$$= \frac{1}{2}\int \mathrm{d}k_a\,e^{ik_at}\,\phi_a^\star(k_a)\,\hat{d}_{k_a}^\dagger \int \mathrm{d}k_b\,e^{ik_at}\,\phi_a^\star(k_b)\,\hat{d}_{k_b}^\dagger\,|0\rangle\,|0\rangle\,. \tag{6.9}$$

If the two photons are indistinguishable, then $\phi_a(k) = \phi_b(k)$ and the first term goes to zero. The state reduces to

$$|\psi\rangle = \frac{1}{2}\left[\left(\Psi_c^\dagger\right)^2 + \left(\Psi_d^\dagger\right)^2\right]|0\rangle\,|0\rangle\,. \tag{6.10}$$

We see that the photons "bunch together" and always appear at the same port.